Higher Minors and Van Kampen's Obstruction

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Abstract

We generalize the notion of graph minors to all (finite) simplicial complexes. For every two simplicial complexes H and K and every nonnegative integer m, we prove that if H is a minor of K then the non vanishing of Van Kampen's obstruction in dimension m (a characteristic class indicating non embeddability in the $(m-1)$-sphere) for H implies its non vanishing for K. As a corollary, based on results by Van Kampen [19] and Flores [5], if K has the d-skeleton of the $(2d+2)$-simplex as a minor, then K is not embeddable in the $2d$-sphere.

We answer affirmatively a problem asked by Dey et. al. [3] concerning topology-preserving edge contractions, and conclude from it the validity of the generalized lower bound inequalities for a special class of triangulated spheres.

1 Introduction

The concept of graph minors has proved be to very fruitful. A famous result by Kuratowski asserts that a graph can be embedded into a 2-sphere iff it contains neither of the graphs K_5 and $K_{3,3}$ as minors. We wish to generalize the notion of graph minors to all (finite) simplicial complexes in a way that would produce analogous statements for embeddability of higher dimensional complexes in higher dimensional spheres. We hope that these higher minors will be of interest in future research, and indicate some results and problems to support this hope.

Let K and K' be simplicial complexes. $K \mapsto K'$ is called an *admissible contraction* if K' is obtained from K by identifying two distinct vertices of K, v and u, such that v and u are not contained in any missing face of K of dimension $\leq dim(K)$. (A set T is called a missing face of K if it is not an element of K while all its proper subsets are.) Specifically, $K' = \{T : u \notin T \in K\} \cup \{(T \setminus \{u\}) \cup \{v\} : u \in T \in K\}$. Note that for graphs (with a nonempty set of edges) this is the usual definition of contraction. An

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equivalent formulation of the condition for admissible contractions is that the following holds:

$$\mathrm{skel}_{\mathrm{dim(K)-2}}(\mathrm{lk(v,K)} \cap \mathrm{lk(u,K)}) = \mathrm{lk(\{v,u\},K)} \tag{1}$$

where $\mathrm{skel_m(K)}$ is the subcomplex of K consisting of faces of dimension $\leq m$.

$K \mapsto K'$ is called a *deletion* if K' is a subcomplex of K. We say that a simplicial complex H is a minor of K, and denote it by $H < K$, if H can be obtained from K by a sequence of admissible contractions and deletions (the relation $<$ is a partial order). Note that for graphs this is the usual notion of a minor.

We now relate this minor notion to Van Kampen's obstruction in cohomology; following Sarkaria [15] we will work with deleted joins and with $\mathbb{Z}_2$ coefficients (background appears in the next section).

Theorem 1.1 *Let* $\mathrm{Sm^m(L)} \in \mathrm{H^m_S(L_*, \mathbb{Z}_2)}$ *denote Van Kampen's obstruction (in equivariant cohomology) for a simplicial complex L, where L_* is the deleted join of L. Let H and K be simplicial complexes. If $H < K$ and* $\mathrm{Sm^m(H)} \neq 0$ *then* $\mathrm{Sm^m(K)} \neq 0$.

For any positive integer d let $H(d)$ be the $(d-1)$-skeleton of the $2d$-dimensional simplex. A well known result by Van Kampen and Flores [5, 19] asserts that the obstruction of $H(d)$ in dimension $(2d-1)$ does not vanish, and hence $H(d)$ is not embeddable in the $2(d-1)$-sphere (note that the case $H(2) = K_5$ is part of the easier direction of Kuratowski's theorem).

Corollary 1.2 *For every $d \geq 1$, if $H(d) < K$ then K is not embeddable in the $2(d-1)$-sphere.* $\square$

Remark: Corollary 1.2 would also follow from the following conjecture:

Conjecture 1.3 *If $H < K$ and K is embeddable in the m-sphere then H is embeddable in the m-sphere.*

Call the following strengthening of (1) the *Link Condition* for the edge $\{u, v\}$:

$$\mathrm{lk(u)} \cap \mathrm{lk(v)} = \mathrm{lk(uv)}. \tag{2}$$

The following theorem answers in the affirmative a question asked by Dey et. al. [3], who already proved the dimension ≤ 3 case.

Theorem 1.4 *Given an edge in a triangulation of a compact PL (piece-wise linear)-manifold without boundary, its contraction results in a PL-homeomorphic space iff it satisfies the Link Condition (2).*

In Section 2 we give the needed background on Van Kampen's obstruction and Smith characteristic class. In Section 3 we prove Theorem 1.1 and show some applications. In Section 4 we prove an analogue of Theorem 1.1 for deleted products over $\mathbb{Z}$. In Section 5 we prove Theorem 1.4 and deduce from it some f-vector consequences. In Section 6 we compare higher minors with graph minors.

2 Algebraic-topological background

The presentation here is based on work of Sarkaria [15, 16] who attributes it to Wu [22] and all the way back to Van Kampen [19]. It is a Smith theoretic interpretation of Van Kampen's obstructions.

Let K be a simplicial complex. The join $K*K$ is the simplicial complex $\{S^1 \uplus T^2 : S, T \in K\}$ (the superscript indicates two disjoint copies of K). The *deleted join* K_* is the subcomplex $\{S^1 \uplus T^2 : S, T \in K, S \cap T = \emptyset\}$. The restriction of the involution $\tau : K*K \longrightarrow K*K$, $\tau(S^1 \cup T^2) = T^1 \cup S^2$ to K_* is into K_*. It induces a $\mathbb{Z}_2$-action on the cochain complex $C^*(K_*;\mathbb{Z}_2)$. For a simplicial cochain complex C over $\mathbb{Z}_2$ with a $\mathbb{Z}_2$-action τ, let C_S be its subcomplex of *symmetric cochains*, $\{c \in C : \tau(c) = c\}$. Restriction induces an action of τ as the identity map on C_S. Note that the following sequence is exact in dimensions ≥ 0:

$$0 \longrightarrow C_S(K_*) \longrightarrow C(K_*) \stackrel{id+\tau}{\longrightarrow} C_S(K_*) \longrightarrow 0$$

where $C_S(K_*) \longrightarrow C(K_*)$ is the trivial injection. (The only part of this statement that may not be true for a general simplicial cochain complex C over $\mathbb{Z}_2$ with a $\mathbb{Z}_2$-action τ, is that $id+\tau$ is surjective.) Thus, there is an induced long exact sequence in cohomology

$$H^0_S(K_*) \stackrel{\mathrm{Sm}}{\longrightarrow} H^1_S(K_*) \longrightarrow ... \longrightarrow H^q_S(K_*) \longrightarrow H^q(K_*) \longrightarrow H^q_S(K_*) \stackrel{\mathrm{Sm}}{\longrightarrow} H^{q+1}_S(K_*) \longrightarrow ....$$

Composing the connecting homomorphism Sm m times we obtain a map $\mathrm{Sm^m : H^0_S(K_*) \longrightarrow H^m_S(K_*)}$. For the fundamental 0-cocycle 1_{K_*}, i.e. the one which maps $\sum_{v\in(K_*)_0} a_v v \mapsto \sum_{v\in(K_*)_0} a_v \in \mathbb{Z}_2$, let $[1_{K_*}]$ denotes its image in $H^0_S(K_*)$. $\mathrm{Sm^m([1_{K_*}])}$ is called the m-th *Smith characteristic class* of K_*, denoted also as $\mathrm{Sm^m(K)}$.

Theorem 2.1 *(Sarkaria [16] Theorem 6.5, see also Wu [22] pp.114-118.) For every $d \geq 1$,* $\mathrm{Sm^{2d-1}(1_{H(d)_*})} \neq 0$.

Theorem 2.2 *(Sarkaria [16] Theorem 6.4 and [15] p.6) If a simplicial complex K embeds in $\mathbb{R}^m$ (or in the m-sphere) then* $\mathrm{Sm^{m+1}(1_{K_*})} = 0$.

Sketch of proof: The definition of Smith class makes sense for singular homology as well; the obvious map from the simplicial chain complex to the singular one induces an isomorphism between the corresponding Smith classes. The definition of deleted join makes sense for subspaces of a Euclidian space as well (see e.g. [13], 5.5); thus an embedding $|K|$ of K into $\mathbb{R}^m$ induces a continuous $\mathbb{Z}_2$-map from $|K|_*$ into the join of $\mathbb{R}^m$ with itself minus the diagonal, which is $\mathbb{Z}_2$-homotopic to the antipodal m-sphere, S^m. The equivariant cohomology of S^m over $\mathbb{Z}_2$ is isomorphic to the ordinary cohomology of $\mathbb{R}P^m$ over $\mathbb{Z}_2$, which vanishes in dimension $m+1$. We get that $\mathrm{Sm^{m+1}(S^m)}$ maps to $\mathrm{Sm^{m+1}(1_{|K|_*})}$ and hence the later equals to zero as well. But $|K_*|$ and $|K|_*$ are $\mathbb{Z}_2$-homotopic, hence $\mathrm{Sm^{m+1}(1_{K_*})} = 0$. $\square$

3 A proof of Theorem 1.1

The idea is to define an injective chain map $\phi : C_*(H;\mathbb{Z}_2) \longrightarrow C_*(K;\mathbb{Z}_2)$ which induces $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{H_*})}$ for every $m \geq 0$.

Lemma 3.1 *Let $K \mapsto K'$ be an admissible contraction. Then it induces an injective chain map $\phi : C_*(K';\mathbb{Z}_2) \longrightarrow C_*(K;\mathbb{Z}_2)$.*

Proof: Fix a labelling of the vertices of K, $v_0, v_1, .., v_n$, such that K' is obtained from K by identifying $v_0 \mapsto v_1$.

Let $F \in K'$. If $F \in K$, define $\phi(F) = F$. If $F \notin K$, define $\phi(F) = \sum\{(F \setminus v) \cup v_0 : v \in F, (F \setminus v) \cup v_0 \in K\}$. Note that if $F \notin K$ then $v_1 \in F$ and $(F \setminus v_1) \cup v_0 \in K$, so the sum above is nonzero. Extend linearly to obtain a map $\phi : C_*(K';\mathbb{Z}_2) \longrightarrow C_*(K;\mathbb{Z}_2)$.

First, let us check that ϕ is a chain map, i.e. that it commutes with the boundary maps ∂. It is enough to verify this for the basis elements F where $F \in K'$. If $F \in K$ then $supp(\partial F) \subseteq K$, hence $\partial(\phi F) = \partial F = \phi(\partial F)$. If $F \notin K$ then $\partial(\phi F) = \partial(\sum\{(F \setminus v) \cup v_0 : v \in F, (F \setminus v) \cup v_0 \in K\})$, and as we work over $\mathbb{Z}_2$, this equals

$$\partial(\phi F) = \sum\{F \setminus v : v \in F, (F \setminus v) \cup v_0 \in K\} + \tag{3}$$

$$\sum\{(F \setminus \{u, v\}) \cup v_0 : u, v \in F, (F \setminus v) \cup v_0 \in K, (F \setminus u) \cup v_0 \notin K\}.$$

On the other hand $\phi(\partial F) = \phi(\sum\{F \setminus u : u \in F, F \setminus u \in K\}) + \phi(\sum\{F \setminus u : u \in F, F \setminus u \notin K\})$ and as we work over $\mathbb{Z}_2$, this equals

$$\phi(\partial F) = \sum\{F \setminus u : u \in F, (F \setminus u) \in K\} + \tag{4}$$

$$\sum\{(F \setminus \{u, v\}) \cup v_0 : u, v \in F, (F \setminus \{u, v\}) \cup v_0 \in K, (F \setminus v) \in K, (F \setminus u) \notin K\}.$$

It suffices to show that in equations (3) and (4) the left summands on the RHSs are equal, as well as the right summands on the RHSs. This follows from observation 3.2 below. Thus ϕ is a chain map.

Second, let us check that ϕ is injective. Let π_K be the restriction map $C_*(K';\mathbb{Z}_2) \longrightarrow \oplus\{\mathbb{Z}_2 F : F \in K' \cap K\}$, $\pi_K(\sum\{\alpha_F F : F \in K'\}) = \sum\{\alpha_F F : F \in K' \cap K\}$. Similarly, let $\pi_K^\perp$ be the restriction map $C_*(K';\mathbb{Z}_2) \longrightarrow \oplus\{\mathbb{Z}_2 F : F \in K' \setminus K\}$. Note that for a chain $c \in C_*(K';\mathbb{Z}_2)$, $c = \pi_K(c) + \pi_K^\perp(c)$ and $supp(\phi(\pi_K(c))) \cap supp(\phi(\pi_K^\perp(c))) = \emptyset$. Assume that $c_1, c_2 \in C_*(K';\mathbb{Z}_2)$ such that $\phi(c_1) = \phi(c_2)$. Then $\pi_K(c_1) = \phi(\pi_K(c_1)) = \phi(\pi_K(c_2)) = \pi_K(c_2)$, and $\phi(\pi_K^\perp(c_1)) = \phi(\pi_K^\perp(c_2))$. Note that for $F_1, F_2 \notin K$, $F_1, F_2 \in K'$ and $F_1 \neq F_2$ $supp(\phi(1F_1)) \ni (F_1 \setminus v_1) \cup v_0 \notin supp(\phi(1F_2))$. Hence also $\pi_K^\perp(c_1) = \pi_K^\perp(c_2)$. Thus $c_1 = c_2$. $\square$

Observation 3.2 *Let $K \mapsto K', v_0 \mapsto v_1$ be an admissible contraction. Let $K' \ni F \notin K$ and $v \in F$. Then $(F \setminus v) \in K$ iff $(F \setminus v) \cup v_0 \in K$.*

Proof: Assume $F \setminus v \in K$. As $(F \setminus v_1) \cup v_0 \in K$ we only need to check the case $v \neq v_1$. We proceed by induction on dim(F). As $\{v_0, v_1\} \in K$ whenever dim(K) > 0, the case dim(F) ≤ 1 is clear. (If dim(K) ≤ 0 there is nothing to prove.) By the induction hypothesis we may assume that all the proper subsets of $(F \setminus v) \cup v_0$ are in K. Also $v_0, v_1 \in (F \setminus v) \cup v_0$. The admissibility of the contraction implies that $(F \setminus v) \cup v_0 \in K$. The other direction is trivial. $\square$

Corollary 3.3 *Let H and K be simplicial complexes. If $H < K$ then there exists an injective chain map $C_*(H; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$.*

Proof: Let the sequence $K = K^0 \mapsto K^1 \mapsto ... \mapsto K^t = H$ demonstrate the fact that $H < K$. If $K^i \mapsto K^{i+1}$ is an admissible contraction, then by Lemma 3.1 it induces an injective chain map $\phi_i : C_*(K^{i+1}; \mathbb{Z}_2) \longrightarrow C_*(K^i; \mathbb{Z}_2)$; and clearly this is also the case if $K^i \mapsto K^{i+1}$ is a deletion - just take ϕ_i to be the obvious injection. Thus, the composition $\phi = \phi_0 \circ ... \circ \phi_{t-1} : C_*(H; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$ is an injective chain map. $\square$

Lemma 3.4 *Let $\phi : C_*(K'; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$ be the injective chain map defined in the proof of Lemma 3.1 for an admissible contraction $K \mapsto K'$. Then for every $m \geq 0$, $\phi^*(\mathrm{Sm}^{\mathrm{m}}([1_{\mathrm{K}_*}])) = \mathrm{Sm}^{\mathrm{m}}([1_{\mathrm{K}'_*}])$ for the induced map ϕ^*.*

Proof: For two simplicial complexes L and L' and a field k, the following map is an isomorphism of chain complexes:

$$\alpha = \alpha_{L,L',k} : C(L;k) \otimes_k C(L';k) \longrightarrow C(L * L';k), \;\; \alpha((1T) \otimes (1T')) = 1(T \uplus T')$$

where $T \in L, T' \in L'$ and α is extended linearly. In case $L = L'$ (in the definition of join we think of L and L' as two disjoint copies of L) and k is understood we denote $\alpha_{L,L',k} = \alpha_L$.

Thus there is an induced chain map $\phi_* : C_*(K' * K'; \mathbb{Z}_2) \longrightarrow C_*(K * K; \mathbb{Z}_2)$, $\phi_* = \alpha_K \circ \phi_\otimes \circ \alpha_{K'}^{-1}$ where $\phi_\otimes : C(K'; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} C(K'; \mathbb{Z}_2) \longrightarrow C(K; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} C(K; \mathbb{Z}_2)$ is defined by $\phi_\otimes(c \otimes c') = \phi(c) \otimes \phi(c')$ (which this is a chain map).

Consider the subcomplex $C_*(K'_*; \mathbb{Z}_2) \subseteq C_*(K' * K'; \mathbb{Z}_2)$. We now verify that every $c \in C_*(K'_*; \mathbb{Z}_2)$ satisfies $\phi_*(c) \in C_*(K_*; \mathbb{Z}_2)$. It is enough to check this for chains of the form $c = 1(S^1 \cup T^2)$ where $S, T \in K'$ and $S \cap T = \emptyset$. For a collection of sets A let $V(A) = \cup_{a \in A} a$. Clearly if the condition

$$V(supp(\phi(S))) \cap V(supp(\phi(T))) = \emptyset \tag{5}$$

is satisfied then we are done. If $v_1 \notin S, v_1 \notin T$, then $\phi(S) = S, \phi(T) = T$ and (5) holds. If $T \ni v_1 \notin S$, then $\phi(S) = S$ and $V(supp\phi(T)) \subseteq T \cup \{v_0\}$.

As $v_0 \notin S$ condition (5) holds. By symmetry, (5) holds when $S \ni v_1 \notin T$ as well.

With abuse of notation (which we will repeat) we denote the above chain map by ϕ, $\phi : C_*(K'_*; \mathbb{Z}_2) \longrightarrow C_*(K_*; \mathbb{Z}_2)$. For a simplicial complex L, the involution $\tau_L : L_* \longrightarrow L_*$, $\tau_L(S^1 \cup T^2) = T^1 \cup S^2$ induces a $\mathbb{Z}_2$-action on $C_*(L_*; \mathbb{Z}_2)$. It is immediate to check that $\alpha_{L,L',k}$ and $\phi_\otimes$ commute with these $\mathbb{Z}_2$-actions, and hence so does their composition, ϕ. Thus, we have proved that $\phi : C_*(K'_*; \mathbb{Z}_2) \longrightarrow C_*(K_*; \mathbb{Z}_2)$ is a $\mathbb{Z}_2$-chain map.

Therefore, there is an induced map on the symmetric cohomology rings $\phi : H^*_S(K_*) \longrightarrow H^*_S(K'_*)$ which commutes with the connecting homomorphisms $\mathrm{Sm} : \mathrm{H^i_S(L)} \longrightarrow \mathrm{H^{i+1}_S(L)}$ for $L = K_*, K'_*$.

Let us check that for the fundamental 0-cocycles $\phi([1_{K_*}]) = [1_{K'_*}]$ holds. A representing cochain is $1_{K_*} : \oplus_{v \in (K_*)_0} \mathbb{Z}_2 v \longrightarrow \mathbb{Z}_2$, $1_{K_*}(1v) = 1$. As $\phi|_{C_0(K'_*)} = id$ (w.r.t. the obvious injection $(K'_*)_0 \longrightarrow (K_*)_0$), for every $u \in (K'_*)_0$ $(\phi 1_{K_*})(u) = 1_{K_*}(\phi|_{C_0(K'_*)}(u)) = 1_{K_*}(u) = 1$, thus $\phi(1_{K_*}) = 1_{K'_*}$.

As ϕ commutes with the Smith connecting homomorphisms, for every $m \geq 0$, $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{K'_*})}$. $\square$

Theorem 3.5 *Let H and K be simplicial complexes. If $H < K$ then there exists an injective chain map $\phi : C_*(H; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$ which induces $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{H_*})}$ for every $m \geq 0$.*

Proof: Let the sequence $K = K^0 \mapsto K^1 \mapsto ... \mapsto K^t = H$ demonstrate the fact that $H < K$. If $K^i \mapsto K^{i+1}$ is an admissible contraction, then by Lemmas 3.1 and 3.4 it induces an injective chain map $\phi_i : C_*(K^{i+1}; \mathbb{Z}_2) \longrightarrow C_*(K^i; \mathbb{Z}_2)$ which in turn induces $\phi_i(\mathrm{Sm^m(1_{(K^i)_*})}) = \mathrm{Sm^m(1_{(K^{i+1})_*})}$ for every $m \geq 0$. If $K^i \mapsto K^{i+1}$ is a deletion - take ϕ_i to be the obvious injection, to obtain the same conclusions. Thus, the composition $\phi = \phi_0 \circ ... \circ \phi_{t-1} : C_*(H; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$ is as desired. $\square$

Proof of Theorem 1.1: By Theorem 3.5 $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{H_*})}$. Thus if $\mathrm{Sm^m(1_{H_*})} \neq 0$ then $\mathrm{Sm^m(1_{K_*})} \neq 0$. $\square$

Example 3.6 *Let K be the simplicial complex spanned by the following collection of* 2*-simplices:* $(\binom{[7]}{3} \setminus \{127, 137, 237\}) \cup \{128, 138, 238, 178, 278, 378\}$.

K is not a subdivision of $H(3)$, and its geometric realization even does not contain a subspace homeomorphic to $H(3)$ (as there are no 7 points in $|K|$, each with a neighborhood whose boundary contains a subspace which is homeomorphic to K_6). Nevertheless, contraction of the edge 78 is admissible and results in $H(3)$. By Theorem 1.1 K has a non-vanishing Van Kampen's obstruction in dimension 5, and hence is not embeddable in the 4-sphere.

Example 3.7 *Let K_1 be a triangulation of S^1 (the 1-sphere) and let K_2 be a triangulation of S^2. Then $K = K_1 * K_2$ is a triangulation of S^4. Let T be a missing triangle of K and $L = \mathrm{skel_2(K) \cup \{T\}}$. Then L does not embed in* $\mathbb{R}^4$.

proof: It is known and easy to prove that every 2-sphere may be reduced to the boundary of the tetrahedron by a sequence of admissible contractions in a way that fixes a chosen triangle from the original triangulation (e.g. [21], Lemma 6). This guarantees the existence of sequences of admissible contractions as described below.

Case 1: $\partial(T) = K_1$. There exists a sequences of admissible contractions (of vertices from K_2) which reduces L to $H(3)$. By Theorems 1.1, 2.1 and 2.2, L does not embed in $\mathbb{R}^4$.

Case 2: $\partial(T) \neq K_1$. Hence $\partial(T) \subseteq K_2$ and separates K_2 into two disks. By performing admissible contractions of pairs of vertices within each of these disks, and within K_1, we can reduce L to the 2-skeleton of the join $L_1 * L_2$ where L_1 is the boundary of a triangle and L_2 is two boundaries of tetrahedra glued along a triangle. Let v be a vertex which belongs to exactly one of the two tetrahedra which were used to define L_2. Deleting v from L results in $H(3)$ minus one triangle which consists of the vertices of L_1. Hence the subcomplex $L' = (L-v) \cup (L_1 * \{v\})$ of L is admissibly contracted into $H(3)$ by contracting an edge which contains v. Thus, $H(3) < L$ and by Theorems 1.1, 2.1 and 2.2, L does not embed in $\mathbb{R}^4$. $\square$

Example 3.7 is a special case of the following conjecture, a work in progress of Uli Wagner and the author.

Conjecture 3.8 *Let K be a triangulated $2d$-sphere and let T be a missing d-face in K. Let $L = \mathrm{skel_d(K)} \cup \{\mathrm{T}\}$. Then L does not embed in $\mathbb{R}^{2d}$.*

4 The obstruction over $\mathbb{Z}$

More commonly in the literature, Van Kampen's obstruction is defined via deleted products and with $\mathbb{Z}$ coefficients, where, except for 2-simplicial complexes, its vanishing is also sufficient for embedding of the complex in a Euclidian space of double its dimension. We obtain an analogue of Theorem 1.1 for this context.

The presentation of the background on the obstruction here is based on the ones in [14], [22] and [18].

Let K be a finite simplicial complex. Its deleted product is $K \times K \setminus \{(x,x) : x \in K\}$, employed with a fixed-point free $\mathbb{Z}_2$-action $\tau(x,y) = (y,x)$. It $\mathbb{Z}_2$-deformation retracts into $K_\times = \cup\{S \times T : S, T \in K, S \cap T = \emptyset\}$, with which we associate a cell chain complex over $\mathbb{Z}$: $C_\bullet(K_\times) = \bigoplus\{\mathbb{Z}(S \times T) : S \times T \in K_\times\}$ with a boundary map $\partial(S \times T) = \partial S \times T + (-1)^{\mathrm{dim S}} S \times \partial T$, where $S \times T$ is a $\mathrm{dim(S \times T)}$-chain. The dual cochain complex consists of the j-cochains $C^j(K_\times) = Hom_{\mathbb{Z}}(C_j(K_\times), \mathbb{Z})$ for every j.

There is a $\mathbb{Z}_2$-action on $C_\bullet(K_\times)$ defined by $\tau(S \times T) = (-1)^{\mathrm{dim(S)dim(T)}} T \times S$. As it commutes with the coboundary map, by restriction of the coboundary map we obtain the subcomplexes of symmetric cochains $C^\bullet_s(K_\times) = \{c \in C^\bullet(K_\times) : \tau(c) = c\}$ and of antisymmetric cochains $C^\bullet_a(K_\times) = \{c \in C^\bullet(K_\times) :$

$\tau(c) = -c\}$. Their cohomology rings are denoted by $H_s^\bullet(K_\times)$ and $H_a^\bullet(K_\times)$ respectively. Let H_{eq}^m be H_s^m for m even and H_a^m for m odd.

For every finite simplicial complex K there is a unique $\mathbb{Z}_2$-map, up to $\mathbb{Z}_2$-homotopy, into the infinite dimensional sphere $i : K_\times \to S^\infty$, and hence a uniquely defined map $i^* : H_{eq}^\bullet(S^\infty) \to H_{eq}^\bullet(K_\times)$. For z a generator of $H_{eq}^m(S^\infty)$ call $o^m = o_{\mathbb{Z}}^m(K_\times) = i^*(z)$ the Van Kampen obstruction; it is uniquely defined up to a sign. It turns out to have the following explicit description: fix a total order $<$ on the vertices of K. It evaluates elementary symmetric chains of even dimension $2m$ by

$$o^{2m}((1+\tau)(S \times T)) = \{^{1 \ if \ s_0<t_0<..<s_m<t_m}_{0 \ otherwise} \tag{6}$$

and evaluates elementary antisymmetric chains of odd dimension $2m+1$ by

$$o^{2m+1}((1-\tau)(S \times T)) = \{^{1 \ if \ t_0<s_0<t_1<..<t_m<s_m<t_{m+1}}_{0 \ otherwise} \tag{7}$$

where the s_l's are elements of S and the t_l's are elements of T. Its importance to embeddability is given in the following classical result:

Theorem 4.1 *[19, 17, 22] If a simplicial complex K embeds in $\mathbb{R}^m$ then $H_{eq}^\bullet(K_\times) \ni o_{\mathbb{Z}}^m(K_\times) = 0$. If K is m-dimensional and $m \neq 2$ then $o_{\mathbb{Z}}^{2m}(K_\times) = 0$ implies that K embeds in $\mathbb{R}^{2m}$.*

In relation to higher minors, the analogue of Theorem 1.1 holds:

Theorem 4.2 *Let H and K be simplicial complexes. If $H < K$ and $o_{\mathbb{Z}}^m(H_\times) \neq 0$ then $o_{\mathbb{Z}}^m(K_\times) \neq 0$.*

From Theorems 4.2 and 4.1 it follows that Conjecture 1.3 is true when $\mathrm{dim(H)} = \mathrm{dim(K)} = \mathrm{m}/2 \neq 2$.

Proof of Theorem 4.2: Fix a total order on the vertices of K, $v_0 < v_1 < .. < v_n$ and consider an admissible contraction $K \mapsto K'$ where K' is obtained from K by identifying $v_0 \mapsto v_1$ (shortly this will be shown to be w.l.o.g.). Define a map ϕ as follows: for $F \in K'$

$$\phi(F) = \{^{F \quad \mathrm{if\ F\in K}}_{\sum\{sgn(v,F)(F\setminus v)\cup v_0:\ v\in F,(F\setminus v)\cup v_0 \in K\} \quad \mathrm{if\ F\notin K}} \tag{8}$$

where $\mathrm{sgn(v,F)} = (-1)^{|\{\mathrm{t\in F:t<v}\}|}$. Extend linearly to obtain an injective $\mathbb{Z}$-chain map $\phi : C_\bullet(K') \longrightarrow C_\bullet(K)$. (The check that this map is indeed an injective $\mathbb{Z}$-chain map is similar to the proof of Lemma 3.1.) In case we contract a general $a \mapsto b$, for the signs to work out consider the map $\tilde{\phi} = \pi^{-1}\phi\pi$ rather than ϕ, where π is induced by a permutation on the vertices which maps $\pi(a) = v_0$, $\pi(b) = v_1$. Then $\tilde{\phi}$ is an injective $\mathbb{Z}$-chain map.

As $\phi(S \times T) := \phi(S) \times \phi(T)$ commutes with the $\mathbb{Z}_2$ action and with the boundary map on the chain complex of the deleted product, ϕ induces a map $H_{eq}^\bullet(K_\times) \to H_{eq}^\bullet(K'_\times)$. It satisfies $\phi^*(o_{\mathbb{Z}}^m(K_\times)) = o_{\mathbb{Z}}^m(K'_\times)$ for all $m \geq 1$. The

checks are straight forward (for proving the last statement, choose a total order with contraction which identifies the minimal two elements $v_0 \mapsto v_1$, and show equality on the level of cochains). We omit the details.

If $K \mapsto K'$ is a deletion, consider the obvious injection $\phi : K' \to K$ to obtain again an induced map with $\phi^*(o_{\mathbb{Z}}^m(K_\times)) = o_{\mathbb{Z}}^m(K'_\times)$.

Let the sequence $K = K^0 \mapsto K^1 \mapsto ... \mapsto K^t = H$ demonstrate the fact that $H < K$. By composing the corresponding maps as above we obtain a map ϕ^* with $\phi^*(o_{\mathbb{Z}}^m(K_\times)) = o_{\mathbb{Z}}^m(H_\times)$ and the result follows. $\square$

Remark [*deleted joins*]: We could define the obstruction over $\mathbb{Z}$ via deleted joins; the relation with the definition via deleted products is verified below.

In K_* fix a total order on the vertices such that $v^1 < u^2$ for every two vertices $v, u \in K$. The $\mathbb{Z}$-simplicial chain complex $C_\bullet(K_*)$ is endowed with a $\mathbb{Z}_2$-action defined by $\tau(S^1 * T^2) = (-1)^{|S||T|} T^1 * S^2$ (check that it commutes with the boundary map $\partial(S^1 * T^2) = \partial(S^1) * T^2 + (-1)^{|S|} S^1 * \partial(T^2)$). Define the equivariant cohomology $H^\bullet_{eq}(K_*)$ via the subcomplexes of symmetric and of antisymmetric cochains, as done for deleted products. The map $f : C_i(K_\times) \to C_{i+1}(K_*)$, $f(S \times T) = (-1)^{|S|} S^1 * T^2$, which shift the dimension by $+1$, is invertible for every $i \geq 0$ and satisfies the relations $\tau \circ f = -f \circ \tau$ and $\partial \circ f = -f \circ \partial$. Hence it induces isomorphisms $H^{i+1}_s(K_*) \to H^i_a(K_\times)$ and $H^{i+1}_a(K_*) \to H^i_s(K_\times)$ for every $i \geq 1$. Denote the pre-image of $o_{\mathbb{Z}}^m(K_\times) \in H^m_{eq}(K_\times)$ in $H^{m+1}_{eq}(K_*)$ by $o_{\mathbb{Z}}^{m+1}(K_*)$. Then the analogues of Theorems 4.1 and 4.2 for $o_{\mathbb{Z}}^{m+1}(K_*)$ clearly hold.

5 Topology preserving edge contractions

5.1 PL manifolds

Proof of Theorem 1.4: Let M be a PL-triangulation of a compact d-manifold without boundary. Let ab be an edge of M and let M' be obtained from M by contracting $a \mapsto b$. We will prove that if the Link Condition (2) holds for ab then M and M' are PL-homeomorphic, and otherwise they are not homeomorphic (not even 'locally homologic'). For $d = 1$ the assertion is clear. Assume $d > 1$. Denote the *closed* star of a vertex a in M by $\mathrm{st(a, M)} = \{\mathrm{T} \in \mathrm{M} : \mathrm{T} \cup \{a\} \in \mathrm{M}\}$ and denote its antistar by $\mathrm{ast(a, M)} = \{\mathrm{T} \in \mathrm{M} : \mathrm{a} \notin \mathrm{T}\}$.

Denote $B(b) = \{b\} * \mathrm{ast(b, lk(a, M))}$ and $L = \mathrm{ast(a, M)} \cap \mathrm{B(b)}$. Then $M' = \mathrm{ast(a, M)} \cup_{\mathrm{L}} \mathrm{B(b)}$. As M is a PL-manifold without boundary, $\mathrm{lk(a, M)}$ is a $(d-1)$-PL-sphere (see e.g. Corollary 1.16 in [7]). By Newman's theorem (e.g. [7], Theorem 1.26) $\mathrm{ast(b, lk(a, M))}$ is a $(d-1)$-PL-ball. Thus $B(b)$ is a d-PL-ball. Observe that $\partial(B(b)) = \mathrm{ast(b, lk(a, M))} \cup \{b\} * \mathrm{lk(b, lk(a, M))} = \mathrm{lk(a, M)} = \partial(\mathrm{st(a, M)})$.

The identity map on $\mathrm{lk(a, M)}$ is a PL-homeomorphism $h : \partial(B(b)) \to \partial(\mathrm{st(a, M)})$, hence it extends to a PL-homeomorphism $\tilde{h} : B(b) \to \mathrm{st(a, M)}$

(see e.g. [7], Lemma 1.21).

Note that $L = \mathrm{lk(a,M)} \cup (\{\mathrm{b}\} * (\mathrm{lk(a,M)} \cap \mathrm{lk(b,M)}))$.

If $\mathrm{lk(a)} \cap \mathrm{lk(b)} = \mathrm{lk(ab)}$ (in M) then $L = \mathrm{lk(a,M)}$, hence gluing together the maps $\tilde{h}$ and the identity map on $\mathrm{ast(a,M)}$ results in a PL-homeomorphism from M' to M.

If $\mathrm{lk(a)} \cap \mathrm{lk(b)} \neq \mathrm{lk(ab)}$ (in M) then $\mathrm{lk(a,M)} \subsetneqq \mathrm{L}$. The case $L = B(b)$ implies that $M' = \mathrm{ast(a,M)}$ and hence M' has a nonempty boundary, showing it is not homeomorphic to M. A small punctured neighborhood of a point in the boundary of M' has trivial homology while all small punctured neighborhoods of points in M has non vanishing $(d-1)$-th homology. This is what we mean by 'not even locally homologic': M and M' have homologically different sets of small punctured neighborhoods.

We are left to deal with the case $\mathrm{lk(a,M)} \subsetneqq \mathrm{L} \subsetneqq \mathrm{B(b)}$. As L is closed there exists a point $t \in L \cap \mathrm{int(B(b))}$ with a small punctured neighborhood $N(t, M')$ which is not contained in L. For a subspace K of M' denote by $N(t, K)$ the neighborhood in K $N(t, M') \cap K$. Thus $N(t, M') = N(t, \mathrm{ast(a,M)}) \cup_{\mathrm{N(t,L)}} \mathrm{N(t,B(b))}$. We get a Mayer-Vietoris exext sequence in reduced homology:

$$H_{d-1}N(t,L) \to H_{d-1}N(t, \mathrm{ast(a,M)}) \oplus \mathrm{H_{d-1}N(t,B(b))} \to \mathrm{H_{d-1}N(t,M')} \to \tag{9}$$

$$H_{d-2}N(t,L) \to H_{d-2}N(t, \mathrm{ast(a,M)}) \oplus \mathrm{H_{d-2}N(t,B(b))}.$$

Note that $N(t, \mathrm{ast(a,M)})$ and $N(t, B(b))$ are homotopic to their boundaries which are $(d-1)$-spheres. Note further that $N(t, L)$ is homotopic to a proper subset X of $\partial(N(t, B(b)))$ such that the pair $(\partial(N(t, B(b))), X)$ is triangulated. By Alexander duality $H_{d-1}N(t, L) = 0$. Thus, (9) simplifies to the exact sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_{d-1}N(t, M') \to H_{d-2}N(t, L) \to 0.$$

Thus, $\mathrm{rank(H_{d-1}N(t,M'))} \geq 2$, hence M and M' are not locally homologic, and in particular are not homeomorphic. □

Remarks: (1) Omitting the assumption in Theorem 1.4 that the boundary is empty makes both implications incorrect. Contracting an edge to a point shows that the Link Condition is not sufficient. Contracting an edge on the boundary of a cone over an empty triangle shows that the Link Condition is not necessary.

(2) The necessity of the Link Condition holds also in the topological category (and not only in the PL category), as the proof of Theorem 1.4 shows. Indeed, for this part we only used the fact that $B(b)$ is a pseudo manifold with boundary $\mathrm{lk(a,M)}$ (not that it is a ball); taking the point t to belong to exactly two facets of $B(b)$. The following part, in the topological category, is still open:

Problem 5.1 *Given an edge in a triangulation of a compact manifold without boundary which satisfies the Link Condition, is it true that its contraction results in a homeomorphic space? Or at least in a space of the same homotopic or homological type?*

A Mayer-Vietoris argument shows that such topological manifolds M and M' have the same Betti numbers; both $\mathrm{st(a, M)}$ and $B(b)$ are cones and hence their homology vanishes.

A candidate for a counterexample for Problem 5.1 may be the join $M = T*P$ where T is the boundary of a triangle and P a triangulation of Poincaré homology 3-sphere, where an edge with one vertex in T and the other in P satisfies the Link Condition. By the double-suspension theorem (Edwards [4] and Cannon [1]) M is a topological 5-sphere.

Walkup [20] mentioned, without details, the necessity of the Link Condition for contractions in topological manifolds, as well as the sufficiency of the Link Condition for the 3 dimensional case (where the category of PL-manifolds coincides with the topological one); see [20], p.82-83.

5.2 PL spheres

Definition 5.2 *Boundary complexes of simplices are* strongly edge decomposable *and, recursively, a triangulated PL-manifold S is* strongly edge decomposable *if it has an edge which satisfies the Link Condition (2) such that both its link and its contraction are strongly edge decomposable.*

By Theorem 1.4 the complexes in Definition 5.2 are all triangulated PL-spheres. Note that every 2-sphere is strongly edge decomposable.

Let vu be an edge in a simplicial complex K which satisfies the Link Condition, whose contraction $u \mapsto v$ results in the simplicial complex K'. Note that the f-polynomials satisfy

$$f(K,t) = f(K',t) + t(1+t)f(\mathrm{lk}(\{\mathrm{vu}\},\mathrm{K}),\mathrm{t}),$$

hence the h-polynomials satisfy

$$h(K,t) = h(K',t) + th(\mathrm{lk}(\{\mathrm{vu}\},\mathrm{K}),\mathrm{t}). \tag{10}$$

We conclude the following:

Corollary 5.3 *The g-vector of strongly edge decomposable triangulated spheres is non negative.* $\square$

Is it also an M-sequence? The strongly edge decomposable spheres (strictly) include the family of triangulated spheres which can be obtained from the boundary of a simplex by repeated Stellar subdivisions (at any face); the later are polytopal, hence their g-vector is an M-sequence. For the case of subdividing only at edges (10) was considered by Gal ([6], Proposition 2.4.3).

6 Graph minors versus higher minors

While Theorem 1.1 is an instance of a property of graph minors which generalizes to higher minors, this is not always the case. Let us mention some properties which do not generalize, and others for which we do not know whether they generalize or not.

- For graphs, if K is a subdivision of H then H is a minor of K. This is not the case for higher minors.

  Example 6.1 *Let H be a triangulated PL 3-sphere whose triangulation contains a knotted triangle $\{12, 23, 13\}$ (e.g. [11] for an example with few vertices and references to Hashimori's first examples. In [8] such spheres were proved to be non-constructible). Then H is a subdivision of $\partial\Delta^4$, the boundary complex of the 4-simplex, but $\partial\Delta^4$ is not a minor of H.*

  Proof: Consider, by contradiction, a sequence of deletions and admissible contractions starting at H and ending at $\partial\Delta^4$. Any deletion would result in a complex with a vanishing 3-homology; further deletions and contractions would keep the 3-homology being zero as they induce the injective chain map from Corollary 3.3. Thus the sequence contains only contractions. Any admissible contraction, assuming we haven't reached $\partial\Delta^4$ yet, must satisfy the Link Condition (2) - as by Alexander duality a sphere can not contain a sphere of the same dimension as a proper subspace. If a contraction $a \mapsto b$ satisfies $a \neq 1, 2, 3$, the PL-homeomorphism constructed in the proof of Theorem 1.4 shows that it results in a PL 3-sphere with $\{12, 23, 13\}$ a knotted triangle. It suffices to show that a contraction where $a \in \{1, 2, 3\}$ also results in a triangulation with a knotted triangle, as this would imply that $\partial\Delta^4$ can never be reached, a contradiction. W.l.o.g. $a = 1$. As $\{12, 23, 13\}$ is knotted in M, the Link Condition implies $b \neq 2, 3$ and $\{b, 2, 3\} \notin M$. Thus $\{b2, 23, b3\}$ is an induced subcomplex in M', and hence there is a deformation retract of $M' - \{b2, 23, b3\}$ onto the induced subcomplex $M'[V(M') - \{b, 2, 3\}] = M[V(M) - \{b, 1, 2, 3\}]$, where $V(K)$ is the set of vertices of a complex K (e.g. [2], Lemma 4). Similarly, $M[V(M) - \{b, 1, 2, 3\}]$ is a deformation retract of $M[V(M) - \{1, 2, 3\}] - \{b\}$. To show that the fundamental group $\pi_1(M' - \{b2, 23, b3\}) \neq 0$ we will show that $\pi_1(M[V(M) - \{1, 2, 3\}] - \{b\}) \neq 0$. We use Van Kampen's theorem for the union $M[V(M) - \{1, 2, 3\}] = (M[V(M) - \{1, 2, 3\}] - \{b\}) \cup int(star(b, M[V(M) - \{1, 2, 3\}]))$: note that the intersection is a deformation retract of $lk(b, M)$ minus the induced subcomplex on $\{1, 2, 3\}$ in it, which is path-connected and simply connected. We conclude

that $\pi_1(M[V(M)-\{1,2,3\}]-\{b\}) \cong \pi_1(M[V(M)-\{1,2,3\}]) \neq 0$, as $\{12, 23, 13\}$ is knotted in M. $\square$

- For a graph K on n vertices, if K has more than $3n-6$ edges then it contains a K_5 minor (Mader proved that it even contains a K_5 subdivision [12]). Is the following generalization to higher minors true?:

  Problem 6.2 *Let $C(d,n))$ be the boundary complex of a cyclic d-polytope on n vertices, and let K be a simplicial complex on n vertices. Does $f_d(K) > f_d(C(2d+1,n))$ imply $H(d+1) < K$?*

  Example 6.3 *Let M_L be the vertex transitive neighborly 4-sphere on 15 vertices* manifold_(4, 15, 5, 1) *found by Frank Lutz [10]. M_L has no universal edges, i.e., every edge is contained in a missing triangle.*

  It is possible that K equals the 2-skeleton of M_L union with a missing triangle would provide a counterexample to Problem 6.2.

- If K is the graph of a triangulated 2-sphere union with a missing edge then it contains a K_5 minor (the condition implies having more than $3n-6$ edges). Is the following generalization to higher minors true?:

  Problem 6.4 *Let K be the union of the d-skeleton of a triangulated $2d$-sphere with a missing d-face. Does $H(d+1) < K$?*

  It is possible that K equals the onion of the 2-skeleton of M_L with a missing triangle would provide a counterexample. But if true, then by Theorems 1.1, 2.1 and 2.2, Conjecture 3.8 will follow.

- A Robertson-Seymour type theorem does not hold for embeddability in higher dimensional spheres:

  Proposition 6.5 *For any $d \geq 2$ There exist infinitely many d-complexes not embeddable in the $2d$-sphere such all of their proper minors do embed in the $2d$-sphere.*

  Proof: By identifying disjoint pair of points, each pair to a point, where each pair lies in the interior of a facet of $H(d+1)$, one obtains topological spaces which are not embeddable in the $2d$-sphere but such that any proper subspace of them is. This was proved by Zaks [23] for $d > 2$ and later by Ummel [18] for $d = 2$. By choosing say m such pairs in each facet, one obtains infinitely many pairwise non-homeomorphic such spaces when m varies. To conclude the claim it suffices to triangulate these spaces in a way that no contraction would be admissible;

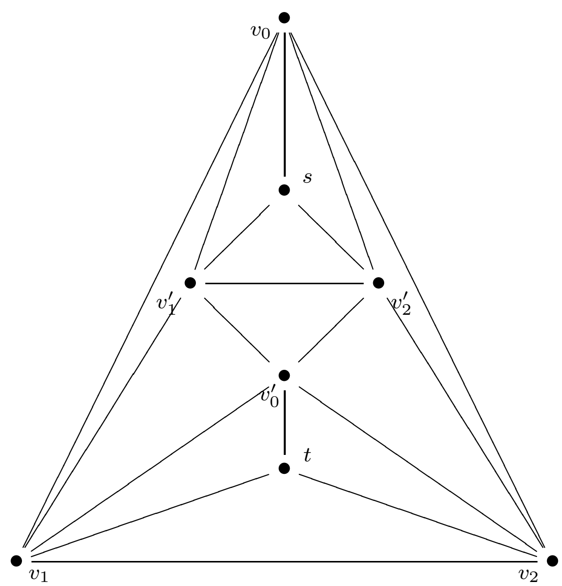

Figure 1: Subdivision of a small facet $F = \{v_0, v_1, v_2\}$.

this is indeed possible (see Figure 6 for an illustration): first subdivide each facet into m small facets say. To identify simplicialy a pair of points s, t in the interior of a small facet $F = \{v_0, ..., v_d\}$ first further subdivide F as follows. Consider the prism $[0,1] \times \{v_1, .., v_d\}$ with bottom $\{v_1, .., v_d\}$ and top $\{v'_1, .., v'_d\}$ and triangulate the cylinder $[0,1] \times \partial\{v_1, .., v_d\}$ without adding new vertices (this is standard). Now cone with a vertex v'_0 over $\partial([0,1] \times \{v_1, .., v_d\})$ to obtain a triangulation of the prism, and further cone with the vertex v_0 over $\partial([0,1] \times \{v_1, .., v_d\}) - \{v_1, .., v_d\}$ to obtain, together with the prism, a triangulation of F. Subdivide $\{v_1, .., v_d, v'_0\}$ by staring from a vertex s in its interior, and subdivide $\{v'_1, .., v'_d, v_0\}$ by staring from a vertex t in its interior. Note that identifying $s \mapsto t$ results in a complex where each pair of vertices from $v_0, .., v_d, v'_0, .., v'_d, t$ is contained in a missing face of dimension $< d$ (a facet for a pair from $v_0, .., v_d$ or from $v'_0, .., v'_d$, and an edge or a triangle with the vertex t for the rest of the pairs). $\square$

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